On Confocal Bicircular Quartics.

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1. *Coordinates.* Throughout this paper the letters x, y will be understood to mean "circular coordinates," viz.

$$x = X + iY = re^{i\vartheta}, \quad y = X - iY = re^{-i\vartheta}, \tag{1}$$

X, Y being rectangular coordinates, and r, ϑ polar coordinates. A third variable z will, wherever convenient, be introduced for homogeneity, and then the definition of x and y will be understood to be modified so as to be given by

$$x : y : z = X + iY : X - iY : 1. \tag{2}$$

The points (x, z), (y, z) are the circular points I, J; the point (x, y) is the origin. It should be observed that a rotation of the axes X, Y through an angle α is equivalent to changing

$$x,\ y \quad \text{into} \quad e^{i\alpha}x,\ e^{-i\alpha}y. \tag{3}$$

2. *The axes of four concyclic points.* It will be desirable to obtain a formula relating to four concyclic points before taking up the consideration of bicircular quartics. If (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) be four points on a circle, the equality of the anharmonic ratios of the pencils through them from I and J respectively may be expressed by the equations

$$\frac{(x_1 - x_2)(x_3 - x_4)}{(y_1 - y_2)(y_3 - y_4)} = \frac{(x_1 - x_3)(x_2 - x_4)}{(y_1 - y_3)(y_2 - y_4)} = \frac{(x_1 - x_4)(x_2 - x_3)}{(y_1 - y_4)(y_2 - y_3)}. \tag{4}$$

Now the value of the first fraction is the clinant* of the pair of lines 12, 34; and likewise for the other fractions. In other words, if we denote by ϑ_{AB} the angle made with the axis of X by the line AB, the common value of the fractions is the value of $e^{2i(\vartheta_{12} + \vartheta_{34})}, = e^{2i(\vartheta_{13} + \vartheta_{24})}, = e^{2i(\vartheta_{14} + \vartheta_{23})}$.

* See this Journal, XII, 162.

Or, what is the same thing, if α be the inclination (i. e. angle with axis of X) of an *axis* of the four points (i. e. of a line parallel to either bisector of any one of the pairs of lines (12, 34), (13, 24), (14, 23)), and if we denote $e^{2i\alpha}$ by Λ, the common value of the fractions is Λ^2.

Suppose, now, that the four points are given by the equations

$$Ax^4 + 4Bx^3z + 6Cx^2z^2 + 4Dxz^3 + Ez^4 = 0, \tag{5}$$

$$A'y^4 + 4B'y^3z + 6C'y^2z^2 + 4D'yz^3 + E'z^4 = 0; \tag{6}$$

the value of Λ^2 may be obtained as follows. Let

$$S = AE - 4BD + 3C^2, \quad T = ACE + 2BCD - AD^2 - EB^2 - C^3,$$
$$S' = A'E' - 4B'D' + 3C'^2, \quad T' = A'C'E' + 2B'C'D' - A'D'^2 - E'B'^2 - C'^3;$$

then, since S/A^2 is a homogeneous quadratic function and T/A^3 a homogeneous cubic function of the numerators of the fractions in (4), and since S'/A'^2 and T'/A'^3 are the like functions of the denominators, it follows that

$$\frac{A'^2 S}{A^2 S'} = (\Lambda^2)^2, \quad \frac{A'^3 T}{A^3 T'} = (\Lambda^2)^3,$$

whence Λ^2 is unambiguously determined. Since the pencils are homographic, we may suppose $S = S'$, $T = T'$; then

$$\Lambda^2 = \frac{A'}{A} \cdot$$

It follows that when the axis of X is taken parallel to an axis of the four points, we have, in addition to $S = S'$ and $T = T'$, $A = A'$.

3. *Confocal bicircular quartics.* The general equation of a bicircular quartic may be written

$$kx^2y^2 + 2xyz(lx + my) + z^2(ax^2 + 2hxy + by^2) + 2z^3(gx + fy) + cz^4 = 0. \tag{7}$$

The system of tangents to this curve from the node (x, z) is found, on writing the equation in the form

$$(kx^2 + 2mxz + bz^2)y^2 + 2(lx^2 + hxz + fz^2)yz + (ax^2 + 2gxz + cz^2)z^2 = 0,$$

to be

$$(ak - l^2)x^4 + 2(am + gk - hl)x^3z + (ab - h^2 + ck - 2fl + 4gm)x^2z^2$$
$$+ 2(bg + cm - fh)xz^3 + (bc - f^2)z^4 = 0, \tag{8}$$

and likewise the system of tangents from the node (y, z) is

$$(bk - m^2)\,y^4 + 2\,(bl + fk - hm)\,y^3z + (ab - h^2 + ck - 2gm + 4fl)\,y^2z^2 \\ + 2\,(af + cl - gh)\,yz^3 + (ac - g^2)\,z^4 = 0. \quad (9)$$

These two pencils are known to be homographic; in point of fact, it is found on trial that the invariants of the quartics (8) and (9) are absolutely equal; and hence the 16 ordinary foci of the curve lie by fours on four circles.

To consider, now, a system of confocal bicircular quartics. The foci may be supposed to be given by the equations

$$(A,\, B,\, C,\, D,\, E\between x,\, z)^4 = 0, \quad (A',\, B',\, C',\, D',\, E'\between y,\, z)^4 = 0, \qquad (10)$$

and these quartics will be supposed so written that $S' = S$, $T' = T$. Then the conditions to which the 9 coefficients of the curve are subjected are obtained by equating the 10 coefficients in (8) and (9) to the 10 coefficients in (10), each multiplied by a common multiplier λ; this gives, on the face of it, 10 equations homogeneous in 10 quantities. But since the coefficients in (8) and (9) satisfy *identically* the relations $S' = S$, $T' = T$, the number of effective equations is reduced to 8; hence there remains an arbitrary parameter; and therefore through any point in the plane there pass a finite number of curves belonging to the system.

How many curves pass through a given point may be determined by considering the curves through the origin, since no restriction has been made on the choice of origin. Putting, then, $c = 0$ and comparing (8) and (9) with (10), we have

$$f^2/g^2 = E/E'. \qquad (11)$$

But $gx + fy = 0$ is the tangent at the origin, and f/g is its clinant; hence there are *at most two directions** in which the curve may pass through the origin; and since the two values of f/g are negatives of each other, these two directions are mutually perpendicular.

If we write equation (11) in the form

$$\frac{A'}{A}\cdot\frac{f^2}{g^2} = \frac{E\,/A}{E'/A'},$$

its full geometrical significance becomes evident. Viz. it is plain that

$$\frac{E}{A} \div \frac{E'}{A'}, = \frac{x_1x_2x_3x_4}{y_1y_2y_3y_4},$$

*The above does not rigorously prove that there actually are two curves through every point, since it has not been shown that both values of f/g are admissible; it does prove that there are *at most* two curves through every point. The equation of the confocal system, which is obtained in the next article, of course determines the matter completely.

is the clinant of the system of rays drawn from the origin to the four foci; and it has been shown in art. 2 that A'/A is the square of the clinant of either axis of the four foci; hence equation (11) signifies that *the angle made with an axis of four concyclic foci of a bicircular quartic by the tangent to the quartic at any point is equal to half the sum of the angles made with the same axis by the four focal radii of the point.**

4. *Equation of the Confocal System.* The foci being given, as before, by equations (10), we may, without loss of geometrical generality, suppose $A' = A$ as well as $S' = S$ and $T' = T$, viz. this is equivalent (art. 3, end) to taking the axes of coordinates parallel to the axes of four concyclic foci. By a proper choice of origin we may effect a further simplification of the problem. Viz. the substitution of $x + \alpha z$ for x and $y + \beta z$ for y does not disturb the equalities already established; and it enables us to make $C' = C$ and $E' = E$. The five relations thus secured may be written as follows:

$$A = A', \quad C = C', \quad E = E', \quad BD = B'D', \quad A(D^2 - D'^2) + E(B^2 - B'^2) = 0.$$

The fourth of these equations will be satisfied if we put $B' = \rho B$ and $D = \rho D'$; or, slightly altering the notation, if we write instead of B, D, B', D' respectively B, ρD, ρB, D; and then the last equation becomes $(1 - \rho^2)(AD^2 - EB^2) = 0$. Thus the equations determining the foci become

$$Ax^4 + 4Bx^3z + 6Cx^2z^2 + 4\rho Dxz^3 + Ez^4 = 0, \qquad (12)$$

$$Ay^4 + 4\rho By^3z + 6Cy^2z^2 + 4Dyz^3 + Ez^4 = 0, \qquad (13)$$

with the relation among the coefficients

$$(1 - \rho^2)(AD^2 - EB^2) = 0. \qquad (14)$$

It will be supposed throughout that neither A nor E vanishes; the vanishing of E would mean that the origin was a focus, the vanishing of A that one of the foci was at infinity.

We next observe that if $\rho = \pm 1$, or if B and D both vanish, four foci lie in a straight line, viz. it is obvious that four points satisfying equations (12) and (13) will in these cases lie either on $x = y$ or on $x = -y$. This is a special case which we shall at first exclude.

* See "Note on the Double Periodicity of the Elliptic Functions," this Journal, XI, 285.

To find the equation of the confocal system, then, we have to subject the general equation of a bicircular quartic

$$kx^2y^2 + 2xyz(lx + my) + z^2(ax^2 + 2hxy + by^2) + 2z^3(gx + fy) + cz^4 = 0$$

to the conditions

$$\left.\begin{array}{llll} \quad(A) & \quad(B) & \quad(C) \\ ak - l^2 = A\lambda, & am + gk - hl = 2B\lambda, & ab - h^2 + ck - 2fl + 4gm = 6C\lambda, \\ bk - m^2 = A\lambda, & bl + fk - hm = 2\rho B\lambda, & ab - h^2 + ck - 2gm + 4fl = 6C\lambda, \\ & \quad(D) & \quad(E) \\ & bg + cm - fh = 2\rho D\lambda, & bc - f^2 = E\lambda, \\ & af + cl - gh = 2D\lambda, & ac - g^2 = E\lambda, \end{array}\right\} \quad (15)$$

where it is to be remembered that either $\rho^2 = 1$ or $AD^2 = EB^2$.

5. *The general case:* $\rho^2 \neq 1$, *B and D not both* 0. Since $AD^2 = EB^2$, and since $A \neq 0$ and $E \neq 0$, neither of the quantities B, D can vanish without the other; therefore neither of them vanishes. From equations (C) we get, by subtraction,

$$fl = gm; \qquad \text{i}$$

then from equations (D) and (E)

$$af^2 - bg^2 = \lambda E(b - a) = 2\lambda D(f - \rho g); \qquad \text{ii}$$

from equations (A) and (B)

$$am^2 - bl^2 = \lambda A(b - a) = 2\lambda B(m - \rho l); \qquad \text{iii}$$

from equations (A) $\qquad l^2 - m^2 = (a - b)k; \qquad$ iv

from equations (E) $\qquad f^2 - g^2 = (a - b)c; \qquad$ v

and from equations (B) and (D)

$$afm - bgl = 2\lambda B(f - \rho g) = 2\lambda D(m - \rho l). \qquad \text{vi}$$

We note that $a \neq b$; for if $b - a$ were 0, equations iv and v would give $l^2 - m^2 = 0$ and $f^2 - g^2 = 0$, while equations ii and iii would give $f - \rho g = 0$ and $m - \rho l = 0$; whence, since $\rho^2 \neq 1$, f, g, l, m must all vanish; but this is impossible, since equations (B) and (D) would then give $B = 0$, $D = 0$.

Since $b-a \neq 0$, it follows from ii and iii that $f-\rho g \neq 0$ and $m-\rho l \neq 0$; but, from i,

$$\frac{f}{m}=\frac{g}{l}=\frac{f-\rho g}{m-\rho l},=\frac{D}{B} \text{ by vi;}$$

so that

$$f=\frac{D}{B}m, \quad g=\frac{D}{B}l. \qquad \text{vii}$$

By iii,

$$b-a=\frac{2B}{A}(m-\rho l), \qquad \text{viii}$$

while ii would give $b-a=\frac{2D}{E}(f-\rho g),=\frac{2D^2}{EB}(m-\rho l)$ by vii; and this is consistent with viii because $AD^2=EB^2$. Equations iv and v (in combination with vii and viii) give

$$k=\frac{A}{2B}\cdot\frac{m^2-l^2}{m-\rho l}, \quad c=\frac{E}{2B}\cdot\frac{m^2-l^2}{m-\rho l}. \qquad \text{ix}$$

From equations (B) we get

$$h=\frac{D}{B}k+\frac{bl-\rho am}{m-\rho l}, \qquad \text{x}$$

while equations D would give $h=\frac{B}{D}c+\frac{bg-\rho af}{f-\rho g}$, which is the same as the value in x because $c=\frac{E}{A}k=\frac{D^2}{B^2}k$ and $\frac{g}{f}=\frac{l}{m}$.

Equations vii, viii, ix and x, together with the equation

$$A(ab-h^2+ck+fl+gm)=3C(ak-l^2+bk-m^2), \qquad \text{xi}$$

[obtained from (A) and (C)] represent *all* of the given equations, the determination of λ being left out of account. Equations vii and ix express f, g, c, k in terms of l and m; it only remains, therefore, to determine a, b and h so as to satisfy viii, x and xi. Since $a-b=(l^2-m^2)/k$ by iv, we may put

$$a=\frac{l^2}{k}+t, \quad b=\frac{m^2}{k}+t,$$

(where it should be noticed that $t \neq 0$, otherwise either of equations (A) would give $A=0$) and then x gives

$$h=\frac{D}{B}k+\frac{lm}{k}+t\frac{l-\rho m}{m-\rho l}.$$

Hence

$$ab - h^2 = t^2 \frac{(1-\rho^2)(m^2-l^2)}{(m-\rho l)^2}$$
$$+ t\left\{\frac{(m+\rho l)(m^2-l^2)}{k(m-\rho l)} - 2k\frac{D}{B}\,\frac{l-\rho m}{m-\rho l}\right\} - k^2\frac{D^2}{B^2} - 2lm\frac{D}{B}.$$

Also

$$k = \frac{A}{2B}\cdot\frac{m^2-l^2}{m-\rho l},\quad ck = \frac{D^2}{B^2}k^2,\quad fl + gm = 2\frac{D}{B}lm, \text{ and } ak - l^2 = bk - m^2 = tk.$$

Making these substitutions, xi becomes

$$At\left\{t\frac{(1-\rho^2)(m^2-l^2)}{(m-\rho l)^2} + \frac{2B}{A}(m+\rho l) - \frac{AD}{B^2}\,\frac{(l-\rho m)(m^2-l^2)}{(m-\rho l)^2}\right\} = \frac{3AC}{B}\cdot\frac{m^2-l^2}{m-\rho l}\cdot t.$$

Rejecting the factor t, this equation gives

$$t = \frac{(m-\rho l)^2}{(1-\rho^2)(m^2-l^2)}\left\{\frac{3C}{B}\cdot\frac{m^2-l^2}{m-\rho l} + \frac{AD}{B^2}\cdot\frac{(l-\rho m)(m^2-l^2)}{(m-\rho l)^2} - \frac{2B}{A}(m+\rho l)\right\}.$$

Also

$$\frac{l^2}{k} = \frac{2B}{A}\cdot\frac{m-\rho l}{m^2-l^2}\cdot l^2 = \frac{(m-\rho l)^2}{(1-\rho^2)(m^2-l^2)}\cdot\frac{2B}{A}\cdot\frac{(1-\rho^2)\,l^2}{m-\rho l}.$$

Hence

$$a, = \frac{l^2}{k} + t,$$
$$= \frac{(m-\rho l)^2}{(1-\rho^2)(m^2-l^2)}\left\{\frac{3C}{B}\cdot\frac{m^2-l^2}{m-\rho l} + \frac{AD}{B^2}\cdot\frac{(l-\rho m)(m^2-l^2)}{(m-\rho l)^2} - \frac{2B}{A}\cdot\frac{m^2-l^2}{m-\rho l}\right\}$$

or
$$(1-\rho^2)\,AB^2a = (3ABC - 2B^3)(m-\rho l) - A^2D\,(\rho m - l);$$

and likewise

$$(1-\rho^2)\,AB^2b = (3ABC - 2\rho^2B^3)(m-\rho l) - A^2D\,(\rho m - l).$$

It conduces to simplicity to express everything in terms of the new parameters α and β defined by

$$m - \rho l = (1-\rho^2)\,B\alpha,\quad \rho m - l = (1-\rho^2)\,B\beta;$$

whence
$$l = B(\rho\alpha - \beta),\qquad m = B(\alpha - \rho\beta).$$

Then the above equations become

$$ABa = (3ABC - 2B^3)\,\alpha - A^2D\beta,$$
$$ABb = (3ABC - 2\rho^2B^3)\,\alpha - A^2D\beta;$$

equations vii and ix become

$$f = D(\alpha - \rho\beta), \quad g = D(\rho\alpha - \beta), \quad k = \frac{A}{2} \cdot \frac{\alpha^2 - \beta^2}{\alpha}, \quad c = \frac{E}{2} \cdot \frac{\alpha^2 - \beta^2}{\alpha};$$

and equation x gives

$$h = \frac{1}{2AB\alpha}\{A^2D(\alpha^2 + \beta^2) - 6ABC\alpha\beta + 4\rho B^3\alpha^2\}.$$

Hence, multiplying each of these values by $2AB\alpha$, and substituting in the equation

$$kx^2y^2 + 2xyz(lx + my) + z^2(ax^2 + 2hxy + by^2) + 2z^3(gx + fy) + cz^4 = 0,$$

we find the equation of the required curves to be

$$\begin{aligned}
& x^2y^2 \{ A^2B\alpha^2 - A^2B\beta^2 \} \\
+ \; & 2x^2yz \{ 2\rho AB^2\alpha^2 - 2AB^2\alpha\beta \} \\
+ \; & 2xy^2z \{ 2AB^2\alpha^2 - 2\rho AB^2\alpha\beta \} \\
+ \; & x^2z^2 \{ (6ABC - 4B^3)\alpha^2 - 2A^2D\alpha\beta \} \\
+ \; & 2xyz^2 \{ (A^2D + 4\rho B^3)\alpha^2 - 6ABC\alpha\beta + A^2D\beta^2 \} \\
+ \; & y^2z^2 \{ (6ABC - 4\rho^2B^3)\alpha^2 - 2A^2D\alpha\beta \} \\
+ \; & 2xz^3 \{ 2\rho ABD\alpha^2 - 2ABD\alpha\beta \} \\
+ \; & 2yz^3 \{ 2ABD\alpha^2 - 2\rho ABD\alpha\beta \} \\
+ \; & z^4 \{ ABE\alpha^2 - ABE\beta^2 \} = 0.
\end{aligned} \tag{16}$$

Let us call this

$$\alpha^2 U - 2\alpha\beta V + \beta^2 W = 0; \tag{17}$$

then U and W are particular curves of the system (being obtained by putting α or $\beta = 0$), but V is not. The equation being quadratic in $\alpha : \beta$, there pass through every point in the plane two curves of the system.

In virtue of the relation $AD^2 = EB^2$, W is a perfect square; viz.

$$W = -A^2B\left(xy - \frac{D}{B}z^2\right)^2, \tag{18}$$

and represents a circle with its centre at the origin, counted twice. And since it is obvious on inspection (bearing in mind that $AD^2 = EB^2$) that the relation $xy = \frac{D}{B}z^2$ converts the y-equation for the foci into the x-equation (equation (12) into equation (13)), this circle is a circle through four foci. Hence *each of the four focal circles, counted twice, is a curve of the system.*

If we put $\alpha = \pm\beta$ (and only so) the term in x^2y^2 disappears, and z becomes

a factor of the equation, so that we have as a limiting case of the quartics a circular cubic together with the line at infinity. But at the same time the term in z^4 disappears; hence the cubic passes through the origin, which has just been seen to be the centre of a focal circle. The terms of lowest degree in x and y are (dropping a constant factor)

$$x \mp y,$$

and the terms of highest degree are

$$xy(x \mp y).$$

Hence, there are two circular cubics confocal with the system of quartics; these cut each other orthogonally at the centre of each of the four focal circles; and the tangent to either cubic at each of these centres is parallel to the asymptote* of that cubic: the tangent and asymptote being in fact parallel to an axis of the foci.

6. *The case of four collinear foci.* If four foci are in a straight line, we may take this line as axis of X, and the centre of gravity of the four foci as origin; then the equations of the I-tangents and the J-tangents are identical: so that in equations (15) [end of art. 4] $\rho = 1$ and $B = 0$. Supposing $D \neq 0$, and noting that in obtaining equations i–vi of art. 5 the relation $AD^2 = EB^2$ (which does not hold in the case now under consideration) was not made use of, we see from iii that $a = b$; then from ii, $f = g$; and then from i, $l = m$. Hence the equations (15) for determining the curve reduce to

$$\left.\begin{aligned} ak - l^2 = A\lambda, \quad (a-h)l + fk = 0, \quad a^2 - h^2 + ck + 2fl = 6C\lambda, \\ (a-h)f + cl = 2D\lambda, \quad ac - f^2 = E\lambda. \end{aligned}\right\} (19)$$

Eliminating h by means of the second of these equations, the other four become

$$ak - l^2 = A\lambda, \qquad \text{i}$$

$$ac - f^2 = E\lambda, \qquad \text{ii}$$

$$-kf^2 + cl^2 = 2Dl\lambda, \qquad \text{iii}$$

$$-k^2f^2 - 2afkl + ckl^2 + 2fl^3 = 6Cl^2\lambda. \qquad \text{iv}$$

iii may be written $k(ac - f^2) - c(ak - l^2) = 2Dl\lambda$, or

$$Ac = Ek - 2Dl; \qquad \text{v}$$

iv may be written $k(-kf^2 + cl^2) - 2fl(ak - l^2) = 6Cl^2\lambda$, or

$$Af = Dk - 3Cl; \qquad \text{vi}$$

* I. e. the asymptote other than the tangents at I and J; the *real* asymptote if the curve is real.

and i, ii, iii obviously give also

$$Af^2 - El^2 = -2Dla. \qquad \text{vii}$$

Equations v and vi give c and f in terms of k and l; substituting for f, equation vii becomes

$$2ADla = -D^2k^2 + 6CDkl + (AE - 9C^2)l^2; \qquad \text{viii}$$

and finally the second of equations (19) gives for h

$$2ADlh = 2ADla + 2ADfk = D^2k^2 + (AE - 9C^2)l^2. \qquad \text{ix}$$

Hence the equation of the required curves, which (since $a = b$, $f = g$, $l = m$) was

$$kx^2y^2 + 2l(x+y)xyz + [a(x^2+y^2) + 2hxy]z^2 + 2f(x+y)z^3 + cz^4 = 0$$

is found to be

$$\begin{array}{rl}
x^2y^2\{ & 2ADkl \hspace{8em} \} \\
+\ 2(x+y)xyz\{ & \hspace{10em} 2ADl^2\} \\
+\ (x^2+y^2)z^2\{ & -D^2k^2 + 6CDkl + (AE - 9C^2)l^2\} \\
+\ 2xyz^2\{ & D^2k^2 \hspace{6em} + (AE - 9C^2)l^2\} \\
+\ 2(x+y)z^3\{ & \hspace{3em} 2D^2kl \hspace{3em} - 6CDl^2\} \\
+\ z^4\{ & \hspace{3em} 2DEkl \hspace{3em} - 4D^2l^2\} = 0,
\end{array} \qquad (20)$$

or

$$-k^2D^2(x-y)^2z^2 + 2klD\{Ax^2y^2 + 3C(x^2+y^2)z^2 + 2D(x+y)z^3 + Ez^4\}$$
$$+ l^2\{4AD(x+y)xyz + (AE - 9C^2)(x+y)^2z^2 - 12CD(x+y)z^3 - 4D^2z^4\} = 0. \qquad (21)$$

7. *Solution of the differential equation* $dx/\sqrt{Ax^4 + 4Bx^3 + 6Cx^2 + 4Dx + E} + dy/\sqrt{Ay^4 + 4By^3 + 6Cy^2 + 4Dy + E} = 0$. In virtue of the property of the tangent to a bicircular quartic contained in the theorem at the end of art. 3, the equation of the system of confocal bicircular quartics whose foci are given by

$$Ax^4 + 4Bx^3 + 6Cx^2 + 4Dx + E = 0, \quad Ay^4 + 4By^3 + 6Cy^2 + 4Dy + E = 0$$

is the solution of the differential equation

$$dx/\sqrt{Ax^4 + 4Bx^3 + 6Cx^2 + 4Dx + E} + dy/\sqrt{Ay^4 + 4By^3 + 6Cy^2 + 4Dy + E} = 0. \qquad (22)$$

In the foregoing article we have obtained the equation of the system when the origin is so chosen that $B = 0$; so that the equation

$$dx'/\sqrt{Ax'^4 + 6C'x'^2 + 4D'x' + E'} + dy'/\sqrt{Ay'^4 + 6C'y'^2 + 4D'y' + E'} = 0 \qquad (23)$$

has for its solution, by (21),

$$-k^2D'^2(x'-y')^2 + 2klD'\{Ax'^2y'^2 + 3C'(x'^2+y'^2) + 2D'(x'+y') + E'\}$$
$$+ l^2\{4AD'(x'+y')x'y' + (AE' - 9C'^2)(x'+y')^2 - 12C'D'(x'+y') - 4D'^2\} = 0, \qquad (24)$$

or say $$-k^2D'^2(x'-y')^2+2klD'P+l^2Q=0. \tag{25}$$

To obtain the solution, then, of (22) we have to transform (25) [which is only an abbreviated expression of (24)] by the substitution

$$x'=x+\frac{B}{A},\quad y'=y+\frac{B}{A}, \tag{26}$$

expressing the result throughout in terms of the coefficients $A \ldots . E$. We shall write

$$u'=Ax'^4+6C'x'^2+4D'x'+E'=Ax^4+4Bx^3+6Cx^2+4Dx+E=u,$$
$$v'=Ay'^4+6C'y'^2+4D'y'+E'=Ay^4+4By^3+6Cy^2+4Dy+E=v.$$

In the first place, then, $x'-y'=x-y$. Secondly

$$\begin{aligned}2P&=u'+v'-A(x'^2-y'^2)^2\\ &=u'+v'-A(x'-y')^2(x'+y')^2\\ &=u+v-A(x-y)^2\left(x+y+2\frac{B}{A}\right)^2\\ &=u+v-A(x^2-y^2)^2-4B(x^3+y^3)+4B(x+y)xy-4\frac{B^2}{A}(x-y)^2\end{aligned}$$

so that

$$P=Ax^2y^2+2B(x+y)xy+3C(x^2+y^2)+2D(x+y)+E-2\frac{B^2}{A}(x-y)^2. \tag{27}$$

It will, then, obviously be advantageous to put $kD'=\alpha-2\frac{B^2}{A}l$, α being a new arbitrary constant; whereupon equation (24) becomes

$$-\alpha^2(x-y)^2+2\alpha l\left[P+2\frac{B^2}{A}(x-y)^2\right]+l^2\left[Q-4\frac{B^2}{A}P-4\frac{B^4}{A^2}(x-y)^2\right]. \tag{28}$$

To transform Q it is convenient to observe that

$$\begin{aligned}\phi,\equiv\ &2(A^2D-3ABC+2B^3)x^3+(A^2E+2ABD-9AC^2+6B^2C)x^2\\ &+2(ABE-3ACD+2B^2D)x+EB^2-AD^2,\end{aligned}$$

is an x-covariant of $(A, B, C, D, E\between x, 1)^4$, i. e. a function which remains unaltered by the transformation $x=x'+\mu$; and the like expression in y will be denoted by ψ. For the quartic $(A, 0, C', D', E'\between x', 1)^4$, ϕ becomes

$$A[2AD'x'^3+(AE'-9C'^2)x'^2-6C'D'x'-D'^2];$$

so that we have

$$\begin{aligned}A^2Q&=2A(\phi+\psi)-A^2(AE'-9C'^2)(x'-y')^2+4A^3D'(x'+y')x'y'-4A^3D'(x'^3+y'^3)\\ &=2A(\phi+\psi)-A^2(x'-y')^2[4AD'(x'+y')+AE'-9C'^2]\\ &=2A(\phi+\psi)-A^2(x-y)^2[4AD'(x+y)+AE'-9C'^2+8BD']\\ &=2A(\phi+\psi)-(x-y)^2[4A(A^2D-3ABC+2B^3)(x+y)\\ &\qquad+A^2(AE-4BD+3C^2)-12(AC-B^2)^2\\ &\qquad+8B(A^2D-3ABC+2B^3)].\end{aligned} \tag{28}$$

Substituting for ϕ and ψ their values, and for P the value found in equation (27), we have

$$Q - 4\frac{B^2}{A}P - 4\frac{B^4}{A^2}(x-y)^2 = -4B^2x^2y^2 + (4AD - 12BC)(x+y)xy$$
$$+ (AE - 9C^2)(x+y)^2 + 8BDxy + (4BE - 12CD)(x+y) - 4D^2,$$

so that the solution of (25) is

$$\begin{aligned} &- \alpha^2(x-y)^2 \\ &+ 2\alpha l\,[Ax^2y^2 + 2B(x+y)xy + 3C(x^2+y^2) + 2D(x+y) + E] \\ &+ l^2\,[-4B^2x^2y^2 + (4AD - 12BC)(x+y)xy + (AE - 9C^2)(x+y)^2 \\ &\qquad + 8BDxy + (4BE - 12CD)(x+y) - 4D^2] = 0. \end{aligned} \quad (29)$$

[Cf. Cayley, Elliptic Functions, p. 339.]

8. *Case of four collinear foci symmetrically situated in respect to their centre of gravity; solution of the differential equation when B and D both vanish.* In art. 6, where the origin was taken so that $B = 0$, it was expressly assumed that D was not also 0; and the equation there found [eq. (21), end of art. 6] ceases to represent a system of curves when in it we put $D = 0$. It is, however, easy to write (21) in such a form that it shall continue to involve an arbitrary constant when $D = 0$; it is also easy to investigate this case independently; but since we have just extended the solution in (21) so as to cover the general case (viz. that in which B is not supposed 0), it will be simplest to obtain the solution for the case $B = 0$, $D = 0$, from the solution for the general case, which is given by equation (29). We thus obtain, for the system of bicircular quartics whose foci are given by

$$Ax^4 + 6Cx^2 + E = 0, \quad Ay^4 + 6Cy^2 + E = 0, \quad (30)$$

the equation

$$-\alpha^2(x-y)^2 + 2\alpha l\,[Ax^2y^2 + 3C(x^2+y^2) + E] + l^2(AE - 9C^2)(x+y)^2 = 0. \quad (31)$$

Putting $l\sqrt{AE - 9C^2} = l'$, $\alpha/\sqrt{AE - 9C^2} = \alpha'$, this may be otherwise written

$$-\alpha'^2(AE - 9C^2)(x-y)^2 + 2\alpha'l'\,[Ax^2y^2 + 3C(x^2+y^2) + E] + l'^2(x+y)^2 = 0. \quad (31')$$

When $AE - 9C^2 = 0$, equation (31) may evidently be written

$$[Axy + 3C + \mu(x-y)][Axy + 3C - \mu(x-y)] = 0, \quad (32)$$

a pair of circles each passing through the two double foci given by $x = y$, $Axy + 3C = 0$; and equation (31′) may be written

$$[Axy - 3C + \mu'(x+y)][Axy - 3C - \mu'(x+y)] = 0, \quad (32')$$

a pair of circles through the two double foci given by $x = -y$, $Axy - 3C = 0$, and orthogonal to the preceding pair.

Equation (31) or (31′) is the solution of the differential equation

$$dx/\sqrt{Ax^4 + 6Cx^2 + E} + dy/\sqrt{Ay^4 + 6Cy^2 + E} = 0;$$

and in the particular case when $AE - 9C^2 = 0$, the two differential equations arising from this, viz.

$$dx/(Ax^2 + 3C) - dy/(Ay^2 + 3C) = 0, \quad dx/(Ax^2 + 3C) + dy/(Ay^2 + 3C) = 0,$$

have for their solution, by (32) and (32′),

$$Axy + 3C = \mu(x - y), \quad Axy - 3C = \mu'(x + y),$$

respectively.

9. *On the solution of the differential equation* $dx/\sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} = Mdy/\sqrt{ay^4 + 4by^3 + 6cy^2 + 4dy + e}$, *the quartics being homographic, and* M *being a certain constant.* Consider, first, the equation

$$\begin{aligned} &dx/\sqrt{Ax^4 + 4Bx^3 + 6Cx^2 + 4\rho Dx + E} \\ &= dy/\sqrt{Ay^4 + 4\rho By^3 + 6Cy^2 + 4Dy + E}. \qquad [AD^2 = EB^2] \end{aligned} \tag{33}$$

This defines a curve in which the inclination of the tangent is half the inclination of the system of rays drawn from its point of contact to a system of four concyclic points determined by

$$(A, B, C, \rho D, E \between x, 1)^4 = 0, \quad (A, \rho B, C, D, E \between y, 1)^4 = 0. \tag{34}$$

Hence the solution of (33) is the equation of the system of bicircular quartics whose foci are given by (34); so that this solution is furnished by equation (16), p. 330, z being therein replaced by 1.

Next, consider the more general equation

$$\begin{aligned} &dx/\sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} \\ &= dy/\sqrt{a'y^4 + 4b'y^3 + 6c'y^2 + 4d'y + e'}. \qquad [S' = S,\ T' = T] \end{aligned} \tag{35}$$

It defines a curve such that the square of the clinant of the tangent at any point is equal to the clinant of the system of rays drawn from the point to a system of four concyclic points determined by

$$(a, b, c, d, e \between x, 1)^4 = 0, \quad (a', b', c', d', e' \between y, 1)^4 = 0, \tag{36}$$

multiplied by a'/a. But, by art. 2, a'/a is the square of the clinant of an axis of these four points; hence, *with respect to an axis of the four points*, the inclina-

tion of the tangent is half the inclination of the system of rays drawn from its point of contact to the four points. The curve defined by (35) is therefore a bicircular quartic belonging to the confocal system whose foci are given by (36). By rotating the axes and moving the origin, i. e. by the transformation

$$x = e^{i\omega}(x_1 + \alpha), \quad y = e^{-i\omega}(y_1 + \beta), \tag{37}$$

equations (36) may be transformed (as shown at the beginning of art. 4) into equations (34); hence the equation of the confocal system just mentioned is given by (16), the x, y, z being replaced by x_1, y_1, 1. This result, converted from an expression in $A, \ldots, E, \rho, x_1, y_1$, into an expression in

$$a, \ldots, e, a', \ldots, e', x, y$$

would furnish the solution of (35).

Finally, if, in the equation

$$dx / \sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} = M dy / \sqrt{a'y^4 + 4b'y^3 + 6c'y^2 + 4d'y + e'},$$

the quartics are homographic, but it is not true that $S' = S$, $T' = T$, we may multiply the second quartic by q, when its invariants will become $S'' = q^2 S'$, $T'' = q^3 T'$; and we may choose q so that

$$q^2 S' = S, \quad q^3 T' = T,$$

whence $q = \dfrac{S'T}{ST'}$. The equation now becomes

$$\begin{aligned} &dx / \sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} \\ &= M\sqrt{\frac{S'T}{ST'}}\, dy / \sqrt{a''y^4 + 4b''y^3 + 6c''y^2 + 4d''y + e''}, \qquad [S'' = S,\ T'' = T] \end{aligned}$$

a'', b'', ... being written for qa', qb', ... Now, if $M = \sqrt{\dfrac{ST'}{S'T}}$, this equation becomes

$$\begin{aligned} &dx / \sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} \\ &= dy / \sqrt{a''y^4 + 4b''y^3 + 6c''y^2 + 4d''y + e''}. \qquad [S'' = S,\ T'' = T] \end{aligned}$$

Thus the solution of the equation

$$dx / \sqrt{ax^4 + 4bx^3 + 6cx^2 + 4dx + e} = \sqrt{\frac{ST'}{S'T}}\, dy / \sqrt{a'y^4 + 4b'y^3 + 6c'y^2 + 4d'y + e'}, \tag{38}$$

the quartics being homographic, is at once reduced to that of equation (35).